If the forces had been $\mu f(r)$ and $\mu F(r')$ we could get the form of the curve by integrating

$$f(r)+F(r')\frac{dr'}{dr}=0.$$

76. Proposed by JAMES F. LAWRENCE, Classical Sophomore, Drury College, Springfield, Mo.

An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the incline plane, x the part of the plane rolled over by the sphere, prove that $(M+m)x'=mx\cos\alpha$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$, where α is the inclination of the plane. [From *Routh's Elementary Rigid Dynamics*, page 126.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let F=friction of the sphere and plane, R=their mutual reaction, θ=the angle through which the sphere has rotated from the beginning of motion, y=the vertical distance of the center of the sphere from the horizontal plane, x_1=the corresponding abscissa, h and k the initial values of x and y, respectively, and a= the radius of the sphere.

For the motion of the sphere, resolving horizontally and vertically, and taking moments about the center of the sphere,

$$m\frac{d^2x_1}{dt^2}=F\cos\alpha-R\sin\alpha\,\ldots\ldots..(1),\quad m\frac{d^2y}{dt^2}=F\sin\alpha+R\cos\alpha-mg\,\ldots\ldots\,..(2),$$

$$\tfrac{2}{5}ma^2\frac{d^2\theta}{dt^2}=aF\,\ldots\ldots..(3).$$

For the horizontal motion of the plane,

$$M\frac{d^2x'}{dt^2}=-F\cos\alpha+R\sin\alpha\,\ldots\ldots..(4).$$

Also, $x_1=h+x'-a\theta\cos\alpha\,\ldots\ldots\ldots\ldots\ldots(5)$, $y=k-a\theta\sin\alpha\,\ldots\ldots\ldots\ldots\ldots(6)$.

From (5) and (1), $m\dfrac{d^2x'}{dt^2}-ma\cos\alpha\dfrac{d^2\theta}{dt^2}=F\cos\alpha-R\sin\alpha\,\ldots\ldots\ldots..(7)$;

and from (6) and (2), $-ma\sin\alpha\dfrac{d^2\theta}{dt^2}=F\sin\alpha+R\cos\alpha-mg\,\ldots\ldots\ldots\ldots..(8)$.

Eliminating F and R from (3), (7) and (8),

$$m\cos\alpha\frac{d^2x'}{dt^2}=\tfrac{7}{5}ma\frac{d^2\theta}{dt^2}-mg\sin\alpha\,\ldots\ldots(9).$$

Integrating (9), noticing that when $t=0$, $\dfrac{dx^2}{dt}=0$, $\dfrac{d\theta}{dt}=0$, and $x'=0$, $\theta=0$,

$$m\cos\alpha.x'=\tfrac{7}{5}ma\theta-\tfrac{1}{2}mg\sin\alpha.t^2 \quad .....(10).$$

Again, eliminating F and R from (1) and (4),

$$m\frac{d^2x_1}{dt^2}+M\frac{d^2x'}{dt^2}=0...(11). \quad \text{But from (5), } m\frac{d^2x_1}{dt^2}=-ma\cos\alpha\frac{d^2\theta}{dt^2}...(12).$$

$$(11)-(12) \text{ gives } ma\theta=\frac{(M+m)x'}{\cos\alpha}......(13).$$

$$(13) \text{ in } (10) \text{ gives } x'=\frac{5m\sin\alpha\cos\alpha}{7(M+m)-5m\cos^2\alpha}.\frac{gt^2}{2}......(14).$$

$$(14) \text{ in } (13) \text{ gives } a\theta=\frac{5(M+m)\sin\alpha}{7(M+m)-5m\cos^2\alpha}.....(15).$$

Since $x=a\theta$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$..(16), and (13) is $(M+m)x'=mx\cos\alpha$..(17).

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, Univeriity of Mississippi, P. O., University, Miss.

The horizontal component of the mutual action between the sphere and the inclined plane imparts to M the acceleration $2x'/t^2$ and to m the acceleration $\frac{2(x\cos\alpha-x')}{t^2}$. The forces producing these opposite motions being equal,

$$M\frac{2x'}{t^2}=m\frac{2(x\cos\alpha-x')}{t^2}.$$

From this $(M+m)x'=mx\cos\alpha$ ...... (1).

The principal of *vis viva* gives

$$2mgx\sin\alpha=M\frac{4x'^2}{t^2}+m\frac{4(x\cos\alpha-x')^2}{t^2}+m\frac{4x^2\sin^2\alpha}{t^2}+mk^2\left(\frac{d\theta}{dt}\right)^2$$

in which the first member is twice the work done by gravity, the first term of the second member has reference to the plane, the second term to the horizontal motion of the sphere, the third to its vertical motion, and the last to its rotation, $d\theta/dt$ being its angular velocity, and k^2 its radius of gyration about a diameter.

The motion of the sphere down the plane being one of pure rolling, $a(d\theta/dt)=2x/t$. (a=radius of sphere).

Substitute $2x/at$ for $d\theta/dt$, and $\frac{2}{5}a^2$ for k^2, and reduce, obtaining

$$\tfrac{1}{2}mgt^2x\sin\alpha=(M+m)x'^2+\tfrac{7}{5}mx^2-2mxx'\cos\alpha.$$

Using (1), this reduces to $\frac{1}{2}gt^2\sin\alpha=\frac{7}{5}x-x'\cos\alpha$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and CHARLES E. MYERS, Canton, O.

Take A as origin. Let (k, l), (y, z) be the coödinates of the center of the sphere when $t=0$ and when $t=t$, respectively, R, the reaction of the plane, P, the friction, a, the radius of the sphere, $\angle DOE=\theta$. Then $AA'=x'$ and $a\theta=x$.

For the sphere the equations of motion are

$\frac{2}{5}ma^2(d^2\theta/dt^2)=aP$..........(1). $m(d^2y/dt^2)=P\cos a-R\sin a$...........(2).

$$m(d^2z/dt^2)=P\sin a+R\cos a-mg\text{..........(3).}$$

For the plane, $M(d^2x'/dt^2)=-P\cos a+R\sin a$.......................(4).

By geometry, $y=k+x'-a\theta\cos a$, $z=l-a\theta\sin a$....................(5, 6).

From (5) and (6) we get $d^2y/dt^2=d^2x'/dt^2-a\cos a(d^2\theta/dt^2)$........(7).

$$d^2z/dt^2=-a\sin a(d^2\theta/dt^2)\text{........(8).}$$

(7)cos$a$+(8)sin$a$ gives

$\cos a(d^2y/dt^2)+\sin a(d^2z/dt^2)=\cos a(d^2x'/dt^2)-a(d^2\theta/dt^2)$...... ....(9).

(2)cos$a$+(3)sin$a$ gives $m\cos a(d^2y/dt^2)+m\sin a(d^2x/dt^2)=P-mg\sin a$......(10).

From (9) and (10), $m\cos a(d^2x'/dt^2)-ma(d^2\theta/dt^2)=P-mg\sin a$.....(11).

(2)+(4) gives $m(d^2y/dt^2)+M(d^2x'/dt^2)=0$................. .....(12).

From (12) and (7) we get $(M+m)(d^2x'/dt^2=ma\cos a(d^2\theta/dt^2)$... ..(13).

The value of P from (1) in (11) gives

$$\tfrac{7}{5}a(d^2\theta/dt^2)-\cos a(d^2x'/dt^2)=g\sin a\text{......(14).}$$

Integrating (13) and (14) and remembering that when $t=0$, $x=0$ and $\theta=0$ we get

$$(M+m)x'=ma\theta\cos a \text{ or } (M+m)x'=mx\cos a,$$

$$\tfrac{7}{5}a\theta-x'\cos a=\tfrac{1}{2}gt\sin a \text{ or } \tfrac{7}{5}x-x'\cos a=\tfrac{1}{2}gt^2\sin a.$$

DIOPHANTINE ANALYSIS.

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71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

I. Solution by CHARLES C. CROSS, Libertytown, Md.

On page 301, Vol. V, of MONTHLY I found four general numbers to be: m, $n^2-1+(m-1)(n-1)^2$, $n(mn+2)$, and $4m(mn^2-mn+2n-1)^2+4(mn^2-mn+2n-1)$.

If we take m and m for the first two numbers, then $m.m+1=\square=y^2$(say) $=(y-s)^2=y^2-2sy+s^2$.

$\therefore y=(s^2+1)/2s$; whence $m=(s^2-1)/2s$.

Let $s=2$, then $m=\frac{3}{4}$. Take $n=2$ and the numbers are $\frac{3}{4}$, $\frac{3}{4}$, $\frac{11}{4}$, 7, and $\frac{315}{4}$.